JOURNAL OF APPROXIMATION THEORY 13, 470–488 (1975)

# Convergence of Padé Approximants to $e^{-z}$ on Unbounded Sets

E. B. SAFF*

*University of South Florida, Tampa, Florida 33620*

AND

R. S. VARGA†

*Kent State University, Kent, Ohio 44242*

DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

## 1. INTRODUCTION

The basic aims of this paper are to study the convergence in the uniform norm of particular Padé approximants to $e^{-z}$ on certain unbounded sets in the complex plane. After some preliminary results are developed in Section 2, we consider in Section 3 the convergence of Padé approximants to $e^{-z}$ on the ray $\{z = x + iy : x \geqslant 0, y = 0\}$. In Theorem 3.1, we give a necessary and sufficient condition for the uniform convergence of a sequence of Padé approximants to $e^{-z}$ on this set, while in Theorem 3.2, we give a sufficient condition for the geometric convergence of a sequence of Padé approximants to $e^{-z}$ on this set. Also, an application of these results to the problem of constrained Chebyshev rational approximations to $e^{-x}$ on $[0, +\infty)$ is included in this section.

In Section 4, the geometric convergence of the particular Padé approximants $\{R_{0,n}(z)\}_{n=0}^{\infty}$ to $e^{-z}$ on unbounded parabolic-like sets in the complex plane is derived in Theorem 4.1, while in Theorem 4.3, it is shown that the particular Padé approximants $\{R_{n-1,n}(z)\}_{n=1}^{\infty}$ and $\{R_{n-2,n}(z)\}_{n=2}^{\infty}$ converge uniformly to $e^{-z}$ on the sectors $S_\delta = \{z = re^{i\theta} : |\theta| \leqslant (\pi/2) - \delta\}$, for any $0 < \delta \leqslant (\pi/2)$.

* Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-73-2503.

† Research supported in part by the U.S. Atomic Energy Commission under Grant AT(11-1)-2075.

## 2. Notation and Preliminary Results

We shall make of the following notation. Let $\pi_m$ denote the set of all complex polynomials in the variable $z$ having degree at most $m$, and let $\pi_{\nu,n}$ denote the set of all complex rational functions $r_{\nu,n}(z)$ of the form

$$r_{\nu,n}(z) = \frac{q_{\nu,n}(z)}{p_{\nu,n}(z)} \text{ where } q_{\nu,n} \in \pi_\nu \,, p_{\nu,n} \in \pi_n \,, p_{\nu,n}(0) = 1.$$

Then, given any function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in a neighborhood of $z = 0$, and given any nonnegative integers $\nu$ and $n$, the $(\nu, n)$-th Padé approximant to $f(z)$ is defined as that element $R_{\nu,n} \in \pi_{\nu,n}$ for which the following expression,

$$f(z) - R_{\nu,n}(z) = \mathcal{O}(|z|^m) \text{ as } |z| \to 0, \tag{2.1}$$

is valid with the largest integer $m$. In the case that $f(z) = e^{-z}$, the $(\nu, n)$-th Padé approximant $R_{\nu,n}(z) \equiv Q_{\nu,n}(z)/P_{\nu,n}(z)$ of $e^{-z}$ is explicitly given by (cf. [12, p. 433; 15, p. 269])

$$Q_{\nu,n}(z) \equiv \sum_{k=0}^{\nu} \frac{(\nu + n - k)!\, \nu!\, (-z)^k}{(\nu + n)!\, k!\, (\nu - k)!}, \tag{2.2}$$

and

$$P_{\nu,n}(z) \equiv \sum_{k=0}^{n} \frac{(\nu + n - k)!\, n!\, z^k}{(\nu + n)!\, k!\, (n - k)!}. \tag{2.3}$$

It is further known that (cf. [11; 12, p. 436; 14]), for finite $z$,

$$\epsilon_{\nu,n}(z) \equiv R_{\nu,n}(z) - e^{-z} = \frac{(-1)^\nu z^{n+\nu+1}}{(n+\nu)!\, e^z P_{\nu,n}(z)} \int_0^1 e^{tz} t^\nu (1-t)^n \, dt. \tag{2.4}$$

In particular, this expression shows that (2.1) is always valid with

$$m = n + \nu + 1,$$

when $f(z) = e^{-z}$. Moreover, as $P_{\nu,n}(x) \geqslant 1$ from (2.3) for all $x \geqslant 0$, it also follows from (2.4) that the error, $\epsilon_{\nu,n}(x)$, for the $(\nu, n)$-th Padé approximant to $e^{-x}$, is of one sign for all $x \geqslant 0$. It is convenient to define the numbers $\eta_{\nu,n}$ as

$$\eta_{\nu,n} \equiv \sup\{|\epsilon_{\nu,n}(x)| : x \geqslant 0\} = \| R_{\nu,n} - e^{-x} \|_{L_\infty[0,\infty]}. \tag{2.5}$$

We begin with

Proposition 2.1. $\eta_{\nu,\nu} = 1$ for all integers $\nu \geqslant 0$.

*Proof.* First, assume $\nu = 2j, j \geqslant 0$. From (2.4), $\epsilon_{2j,2j}(x) \geqslant 0$ for all $x \geqslant 0$. Next, it is clear upon comparing coefficients in (2.2)-(2.3) that $R_{2j,2j}(x) \leqslant 1$ for all $x \geqslant 0$, with $R_{2j,2j}(x) \to 1$ as $x \to +\infty$. Hence,

$$0 \leqslant \epsilon_{2j,2j}(x) = R_{2j,2j}(x) - e^{-x} \leqslant 1 - e^{-x} \leqslant 1 \text{ for all } x \geqslant 0,$$

with $\epsilon_{2j,2j}(x) \to 1$ as $x \to +\infty$. Thus it follows that $\eta_{2j,2j} = 1$.

Assuming $\nu = 2j + 1, j \geqslant 0$, note that $Q_{2j+1,2j+1}(x) = P_{2j+1,2j+1}(-x)$ for any real $x$. Thus, we can write $R_{2j+1,2j+1}(x) = P_{2j+1,2j+1}(-x)/P_{2j+1,2j+1}(x)$, or equivalently,

$$R_{2j+1,2j+1}(x) = \frac{\{P_{2j+1,2j+1}(x) + P_{2j+1,2j+1}(-x)\}}{P_{2j+1,2j+1}(x)} - 1 \text{ for all } x.$$

But from (2.3), $\{P_{2j+1,2j+1}(x) + P_{2j+1,2j+1}(-x)\} \equiv \tilde{P}_{2j}(x)$, a polynomial of degree $2j$, is a positive sum of even powers of $x$ with constant term 2, so that trivially $\tilde{P}_{2j}(x) \geqslant 1$ for all $x \geqslant 0$. Next, by comparing coefficients, it is easy to verify from (2.3) that $P_{2j+1,2j+1}(x) \leqslant e^x$ for all $x \geqslant 0$. Thus, from (2.4),

$$0 \leqslant -\epsilon_{2j+1,2j+1}(x) = e^{-x} - R_{2j+1,2j+1}(x) = e^{-x} - \frac{\tilde{P}_{2j}(x)}{P_{2j+1,2j+1}(x)} + 1$$
$$\leqslant e^{-x} - \frac{1}{e^x} + 1 = 1, \text{ for } x \geqslant 0,$$

with $-\epsilon_{2j+1,2j+1}(x) \to 1$ as $x \to \infty$. Thus, $\eta_{2j+1,2j+1} = 1$. Q.E.D.

We now state an identity in (2.6), which can be obtained by directly appealing to the definitions of $Q_{\nu,n}$ and $P_{\nu,n}$ in (2.2) and (2.3).

LEMMA 2.2. *For any* $\nu \geqslant 0, n \geqslant 1$,

$$\frac{d}{dx}[e^x Q_{\nu,n}(x) - P_{\nu,n}(x)] = \frac{n}{(n+\nu)}[e^x Q_{\nu,n-1}(x) - P_{\nu,n-1}(x)]. \quad (2.6)$$

With these results and with the definition of $\eta_{\nu,n}$ in (2.5). we now prove

THEOREM 2.3. *For any nonnegative integers* $\nu$ *and* $n$ *with* $n > \nu$,

$$\eta_{\nu,n} \leqslant \frac{n}{(2n+\nu)}\eta_{\nu,n-1}. \quad (2.7)$$

*Thus,*

$$\eta_{\nu,n} \leqslant \prod_{j=1}^{n-\nu}\left(\frac{\nu+j}{3\nu+2j}\right) \leqslant \frac{1}{2^{n-\nu}}, \qquad \text{for all} \quad 0 \leqslant \nu < n. \quad (2.8)$$

*Proof.* Using (2.4), we see that $(-1)^\nu \epsilon_{\nu,n}(x) \geqslant 0$ for all $x \geqslant 0$. Because $n$ exceeds $\nu$ by hypothesis then $\epsilon_{\nu,n}(x) \to 0$ as $x \to +\infty$. Hence, there exists a $\xi > 0$ for which $(-1)^\nu \epsilon_{\nu,n}(\xi) = \eta_{\nu,n}$, and $\epsilon'_{\nu,n}(\xi) = 0$. Now, from (2.4), we can write

$$(-1)^\nu \epsilon_{\nu,n}(x) \cdot e^x \cdot P_{\nu,n}(x) = (-1)^\nu\{e^x Q_{\nu,n}(x) - P_{\nu,n}(x)\}.$$

Thus, on differentiating the above expression and evaluating the result at $x = \xi$, we obtain since $\epsilon'_{\nu,n}(\xi) = 0$ that

$$(-1)^\nu \epsilon_{\nu,n}(\xi)\, e^\xi[P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] = (-1)^\nu \frac{d}{dx}\, [e^x Q_{\nu,n}(x) - P_{\nu,n}(x)]|_{x=\xi}\,.$$

Hence, from (2.6) of Lemma 2.2,

$$(-1)^\nu \epsilon_{\nu,n}(\xi)\, e^\xi[P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] = \frac{(-1)^\nu n}{(n+\nu)}\, [e^\xi Q_{\nu,n-1}(\xi) - P_{\nu,n-1}(\xi)]. \quad (2.9)$$

Now, from (2.3), it is easy to verify that $P'_{\nu,n}(x) = (n/(n+\nu))\, P_{\nu,n-1}(x)$ for all $x$, and that $P_{\nu,n}(x) \geqslant P_{\nu,n-1}(x)$ for all $x \geqslant 0$. Thus,

$$(-1)^\nu \epsilon_{\nu,n}(\xi)\, e^\xi[P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] \geqslant \left(\frac{2n+\nu}{n+\nu}\right) (-1)^\nu \epsilon_{\nu,n}(\xi)\, e^\xi P_{\nu,n-1}(\xi).$$

Using (2.9), this implies that

$$\frac{(-1)^\nu n}{(n+\nu)}\, [e^\xi Q_{\nu,n-1}(\xi) - P_{\nu,n-1}(\xi)] \geqslant \left(\frac{2n+\nu}{n+\nu}\right) (-1)^\nu \epsilon_{\nu,n}(\xi)\, e^\xi P_{\nu,n-1}(\xi),$$

or

$$0 \leqslant (-1)^\nu \epsilon_{\nu,n}(\xi) \leqslant \frac{(-1)^\nu n}{(2n+\nu)}\, [R_{\nu,n-1}(\xi) - e^{-\xi}] = \frac{(-1)^\nu n}{(2n+\nu)}\, \epsilon_{\nu,n-1}(\xi).$$

Since $(-1)^\nu \epsilon_{\nu,n}(\xi) = \eta_{\nu,n}$ and since $(-1)^\nu \epsilon_{\nu,n-1}(\xi) \leqslant \eta_{\nu,n-1}$, then

$$\eta_{\nu,n} \leqslant \left(\frac{n}{2n+\nu}\right) \eta_{\nu,n-1}\,,$$

the desired result of (2.7). By induction on the above inequality, it follows that

$$\eta_{\nu,n} \leqslant \left\{ \prod_{j=1}^{n-\nu} \left(\frac{\nu+j}{3\nu+2j}\right)\right\} \eta_{\nu,\nu} = \prod_{j=1}^{n-\nu} \left(\frac{\nu+j}{3\nu+2j}\right), \quad (2.10)$$

the last expression following from Proposition 2.1. But as each term in the above product is at most $\frac{1}{2}$, then we obtain the desired result of (2.8). Q.E.D.

We remark that the inequality of (2.8) reduces in the case $\nu = 0$ to

$$\eta_{0,n} \leqslant 1/2^n,$$

which was first established in Cody, Meinardus, and Varga [2].

For the special case $\nu = n - 1$, we note that the inequality of (2.7), coupled with Proposition 2.1, gives simply

$$\eta_{n-1,n} \leqslant \left(\frac{n}{3n-1}\right), \qquad \text{for all} \quad n \geqslant 1,$$

which implies only the boundedness of the sequence $\{\eta_{n-1,n}\}_{n=1}^{\infty}$. Actually, for our later use in Theorem 3.1, we need that $\eta_{n-1,n}$ tends to zero as $n \to \infty$, but we prove the following stronger result.

**Proposition** 2.4. *There exist positive constants $A_1$ and $A_2$ such that*

$$\frac{A_1}{n} \leqslant \eta_{n-1,n} \leqslant \frac{A_2 \ln n}{n} \qquad \text{for all} \quad n > 1. \tag{2.11}$$

*Proof.* With definitions in (2.2) and (2.3), it is easy to show, by comparing coefficients, that

$$|R_{n-1,n}(x)| = \left|\frac{Q_{n-1,n}(x)}{P_{n-1,n}(x)}\right| \leqslant \frac{Q_{n-1,n}(-x)}{P_{n-1,n}(x)} \leqslant \left(1 + \frac{x}{2n-1}\right)^{-1}$$

for all $x \geqslant 0$, $n \geqslant 1$. Thus, from (2.4),

$$|\epsilon_{n-1,n}(x)| \leqslant |R_{n-1,n}(x)| + e^{-x} \leqslant e^{-x} + \left(1 + \frac{x}{2n-1}\right)^{-1}, \quad x \geqslant 0. \tag{2.12}$$

On the other hand, the integral representation in (2.4) gives us that

$$|\epsilon_{n-1,n}(x)| = \frac{x^{2n}}{(2n-1)!\, e^x P_{n-1,n}(x)} \int_0^1 e^{tx} t^{n-1} (1-t)^n \, dt, \quad x \geqslant 0,$$

which can be written in the form

$$|\epsilon_{n-1,n}(x)| = \frac{x^{2n}}{(2n-1)!\, P_{n-1,n}(x)} \int_0^1 e^{-tx} t^n (1-t)^{n-1} \, dt, \qquad x \geqslant 0.$$

A simple calculation shows that the above integrand, considered as a function of $t \in [0, 1]$ is maximized when $t = u_n(x)$, where

$$0 < u_n(x) \equiv \frac{2n}{(2n-1+x) + ((2n-1+x)^2 - 4nx)^{1/2}} < 1, \qquad n > 1. \tag{2.13}$$

Thus, $|\epsilon_{n-1,n}(x)|$ can be bounded above by

$$|\epsilon_{n-1,n}(x)| \leqslant \frac{x^{2n}(u_n(x))^n (1-u_n(x))^{n-1}}{(2n-1)!\, e^{x\cdot u_n(x)} P_{n-1,n}(x)}, \qquad x \geqslant 0.$$

Next, it follows from (2.3) that $P_{n-1,n}(x) \geqslant ((n-1)!\, x^n/(2n-1)!)$ for all $x \geqslant 0$, so that

$$|\epsilon_{n-1,n}(x)| \leqslant \frac{[x \cdot u_n(x)]^n (1-u_n(x))^{n-1}}{(n-1)!\, e^{x\cdot u_n(x)}}, \qquad x \geqslant 0,$$

and since $e^{x\cdot u_n(x)} \geqslant (x \cdot u_n(x))^n/n!$ and since $e^{-u_n(x)} \geqslant 1 - u_n(x) > 0$, then the above inequality implies that

$$|\epsilon_{n-1,n}(x)| \leqslant ne^{-(n-1)\cdot u_n(x)}, \qquad x \geqslant 0. \tag{2.14}$$

Consequently, from (2.12) and (2.14),

$$|\epsilon_{n-1,n}(x)| \leqslant \min \left\{ne^{-(n-1)\cdot u_n(x)};\, e^{-x} + \left(1 + \frac{x}{2n-1}\right)^{-1}\right\}, \quad x \geqslant 0. \tag{2.15}$$

Now, let $\alpha_n = n^2/(6 \ln n)$, $n > 1$. For all $x \geqslant \alpha_n$, it is clear that

$$e^{-x} + \left(1 + \frac{x}{2n-1}\right)^{-1} \leqslant e^{-\alpha_n} + \left(1 + \frac{\alpha_n}{2n-1}\right)^{-1} \leqslant A\,\frac{\ln n}{n}, \qquad x \geqslant \alpha_n, \tag{2.16}$$

for some positive constant $A$ independent of $n$. Next, using (2.13), for $0 \leqslant x \leqslant \alpha_n$,

$$u_n(x) \geqslant \frac{2n}{2(2n-1+x)} \geqslant \frac{n}{2n-1+\alpha_n} \geqslant \frac{3 \ln n}{n-1},$$

for all $n$ sufficiently large. Hence,

$$ne^{-(n-1)u_n(x)} \leqslant ne^{-3 \ln n} = \frac{1}{n^2} \qquad \text{for} \quad 0 \leqslant x \leqslant \alpha_n. \tag{2.17}$$

Consequently, using (2.15)–(2.17) and the definition of $\eta_{\nu,n}$ in (2.5), then

$$\eta_{n-1,n} \leqslant A_2(\ln n)/n$$

for all $n > 1$.

To obtain the first inequality of (2.11), we first write $P_{n-1,n}(x)$ in the form

$$P_{n-1,n}(x) = \frac{n!}{(2n-1)!}\, x^n \sum_{m=0}^{n} \frac{(n-1+m)!}{(n-m)!\, m!\, x^m}, \qquad x \neq 0.$$

Because

$$\frac{(n-1+m)!}{(n-m)!\,m!} = \frac{n(n^2-1)\cdots[n^2-(m-1)^2]}{m!} \leqslant \frac{n^{2m-1}}{m!}, \qquad 0 \leqslant m \leqslant n,$$

the above sum can be bounded above by

$$P_{n-1,n}(x) \leqslant \frac{n!\,x^n}{n\cdot(2n-1)!}\sum_{m=0}^{n}\frac{(n^2/x)^m}{m!} < \frac{n!\,x^n}{n(2n-1)!}\,e^{n^2/x}, \qquad x>0.$$

Now, let $x = 2n^2$. Since $e^{1/2} < 2$, this implies that

$$P_{n-1,n}(2n^2) < \frac{n!\,2^{n+1}n^{2n-1}}{(2n-1)!}. \tag{2.18}$$

To obtain a similar lower bound for $Q_{n-1,n}(x)$, we first write $Q_{n-1,n}(x)$ in the form

$$(-1)^{n-1}\,Q_{n-1,n}(x) = \frac{(n-1)!\,x^{n-1}}{(2n-1)!}\sum_{m=0}^{n-1}\frac{(n+m)!\,(-1)^m}{m!\,(n-1-m)!\,x^m}, \qquad x \neq 0.$$

For $x = 2n^2$, the above sum is an alternating sum with strictly decreasing terms, so that $(-1)^{n-1}\,Q_{n-1,n}(2n^2)$ exceeds the sum of the first two terms:

$$(-1)^{n-1}\,Q_{n-1,n}(2n^2) > \frac{(n-1)!\,(2n^2)^{n-1}\cdot(n^2+1)}{2n\cdot(2n-1)!}. \tag{2.19}$$

Thus, from (2.18) and (2.19),

$$\frac{(-1)^{n-1}\,Q_{n-1,n}(2n^2)}{P_{n-1,n}(2n^2)} > \frac{(n^2+1)}{8n^3},$$

so that $(-1)^{n-1}\,\epsilon_{n-1,n}(2n^2) > (n^2+1)/8n^3 - (-1)^{n-1}\,e^{-2n^2}$. It is thus clear that

$$(-1)^{n-1}\,\epsilon_{n-1,n}(2n^2) \geqslant \frac{A_1}{n},$$

which implies the first inequality of (2.11). Q.E.D.

## 3. The Convergence of Padé Approximants to $e^{-x}$ on $[0, +\infty)$

Based on the results of the previous section, we now establish the convergence of particular Padé approximants to $e^{-x}$ on the infinite segment $[0, +\infty)$. Actually, we are interested in two kinds of convergence, namely, the *uniform* convergence and, more particularly, the *geometric* convergence of sequences of Padé approximants to $e^{-x}$ on $[0, +\infty)$. We first treat uniform convergence in

THEOREM 3.1. *The sequence* $\{R_{\nu(n),n}\}_{n=1}^{\infty}$ *of Padé approximants converges uniformly to* $e^{-x}$ *on* $[0, \infty)$ *if and only if* $\nu(n) < n$ *for all n sufficiently large.*

*Proof.* Assume first that $\nu(n) < n$ for all $n \geqslant n_0$. From (2.7) and (2.8), we have that

$$\eta_{\nu(n),n} \leqslant \eta_{\nu(n),\nu(n)+1}, \qquad n \geqslant n_0, \tag{3.1}$$

and that

$$\eta_{\nu(n),n} \leqslant \frac{1}{2^{n-\nu(n)}}, \qquad n \geqslant n_0. \tag{3.2}$$

How, given any $\epsilon > 0$, there is, from Proposition 2.4, an $n_1(\epsilon)$ such that $\eta_{n,n+1} < \epsilon$ for all $n > n_1(\epsilon)$. We may assume that $n_1(\epsilon) \geqslant n_0$. Next, choose $n_2(\epsilon) > n_1(\epsilon)$ such that $2^{-n_2(\epsilon)+n_1(\epsilon)} < \epsilon$. Consider then any $n \geqslant n_2(\epsilon)$. If $0 \leqslant \nu(n) \leqslant n_1(\epsilon)$, then using (3.2),

$$\eta_{\nu(n),n} \leqslant \frac{1}{2^{n-\nu(n)}} \leqslant \frac{1}{2^{n_2(\epsilon)-n_1(\epsilon)}} < \epsilon.$$

On the other hand, suppose that $n_1(\epsilon) < \nu(n) \leqslant n - 1$. With the inequality of (3.1) and the fact that $\eta_{n,n+1} < \epsilon$ for all $n > n_1(\epsilon)$, then $\eta_{\nu(n),n} < \epsilon$. Thus, for any $n \geqslant n_2(\epsilon)$ and for any $\nu(n)$ with $\nu(n) < n$, we have that $\eta_{\nu(n),n} < \epsilon$.

Conversely, assume that $\{\eta_{\nu(n),n}\}_{n=1}^{\infty}$ converges to zero as $n \to \infty$. Since $\eta_{\nu,n}$ is finite only if $\nu \leqslant n$, and since $\eta_{\nu,\nu} = 1$ for all $\nu \geqslant 0$ from Proposition 2.1, then evidently $\nu(n) < n$ for all $n$ sufficiently large. Q.E.D.

To establish a sufficient condition for the geometric convergence of certain Padé approximants to $e^{-x}$ on $[0, +\infty)$, we need only use (2.8) of Theorem 2.3 to prove

THEOREM 3.2. *If* $\limsup_{n\to\infty} \{\prod_{j=1}^{n-\nu(n)} (\nu(n)+j)/(3\nu(n)+2j)\}^{1/n} = \alpha < 1$, *then the sequence of Padé approximants* $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ *converges geometrically in the uniform norm to* $e^{-x}$ *on* $[0, +\infty)$, *i.e.*,

$$\limsup_{n\to\infty} (\eta_{\nu(n),n})^{1/n} \leqslant \alpha < 1. \tag{3.3}$$

*As a special case, if* $\limsup_{n\to\infty} (\nu(n)/n) = \beta < 1$, *then*

$$\limsup_{n\to\infty} (\eta_{\nu(n),n})^{1/n} \leqslant \frac{1}{2^{1-\beta}} < 1. \tag{3.4}$$

While the result of Theorem 3.2 establishes a sufficient condition for the geometric convergence of the Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ to $e^{-x}$ on $[0, +\infty)$, it is not known whether this condition is also necessary. On the other hand, from the lower bound in Proposition 2.4, i.e.,

$$\frac{A_1}{n} \leqslant \eta_{n-1,n}, \qquad \text{for all} \quad n \geqslant 1,$$

it is clear that the particular Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ with $\nu(n) = n - 1$, for which $\limsup_{n\to\infty} (\nu(n)/n) = 1$, *cannot* possess geometric convergence to $e^{-x}$ on $[0, +\infty)$. More generally, it can be shown that *no* sequence of the form $\{R_{n-\mu,n}\}_{n=\mu}^{\infty}$, $\mu \geqslant 1$ fixed, converges geometrically on this ray.

It is interesting to note that the result of Theorem 3.2 has applications to the problem of constrained Chebyshev rational approximations to $e^{-x}$ on $[0, +\infty)$. We use the following notation. Let $\hat{\pi}_m$ be the set of all *real polynomials of degree at most m, let* $\hat{\pi}_{\nu,n}$ be the set of all *real* rational functions $r_{\nu,n}(x)$ of the form $r_{\nu,n}(x) = q_{\nu,n}(x)/p_{\nu,n}(x)$, where $q_{\nu,n} \in \hat{\pi}_\nu$, and $p_{\nu,n} \in \hat{\pi}_n$, and $p_{\nu,n}(0) = 1$, and, for any nonnegative integer $k$ with $0 \leqslant k \leqslant n + \nu + 1$, let $\hat{\pi}_{\nu,n}^{(k)}$ be the subset of those $r_{\nu,n}$ in $\hat{\pi}_{\nu,n}$ for which

$$e^{-x} - r_{\nu,n}(x) = \mathcal{O}(|x|^k), \ x \text{ real}, \ |x| \to 0.$$

Then, for any nonnegative integers $n$, $\nu$, and $k$ with $0 \leqslant \nu \leqslant n$ and with $0 \leqslant k \leqslant n + \nu + 1$, the *constrained Chebyshev constants* $\lambda_{\nu,n}^{(k)}$ for $e^{-x}$ on $[0, +\infty)$ are defined as

$$\lambda_{\nu,n}^{(k)} = \inf\{\sup_{0\leqslant x<\infty} |e^{-x} - r_{\nu,n}(x)| : r_{\nu,n} \in \hat{\pi}_{\nu,n}^{(k)}\}. \tag{3.5}$$

For the special case $k = 0$, these (unconstrained) Chebyshev constants for $e^{-x}$ have been studied in [2], Newman [8], and Schönhage [13]. Note that because the $(\nu, n)$-the Padé approximant $R_{\nu,n}$ is real, i.e., $R_{\nu,n} \in \hat{\pi}_{\nu,n}$, the special case $k = n + \nu + 1$ is, from (2.5) such that $\lambda_{\nu,n}^{(n+\nu+1)} = \eta_{\nu,n}$.

Recently, J. D. Lawson [5] has considered the particular constrained Chebyshev constants $\lambda_{n,n}^{(n+1)}$ for $e^{-x}$ on $[0, +\infty)$, and, from his computed values of $\lambda_{n,n}^{(n+1)}$ for $2 \leqslant n \leqslant 5$, one would naturally suspect the geometric convergence of these constants to zero. That this is theoretically so can be seen to be a special case of

**Theorem 3.3.** *Assume that the sequence of nonnegative integers*

$$\{k(n)\}_{n=0}^{\infty},$$

*satisfying* $0 \leqslant k(n) \leqslant 2n + 1$ *for every* $n \geqslant 0$, *has the property that*

$$\limsup_{n\to\infty} \left(\frac{k(n) - (n+1)}{n}\right) = \alpha < 1, \tag{3.6}$$

*and define* $\delta = \max(0, \alpha)$. *Then, for any sequence of nonnegative integers* $\{m(n)\}_{n=0}^{\infty}$ *satisfying* $\max\{0; k(n) - (n+1)\} \leqslant m(n) \leqslant n$,

$$\frac{1}{1280} \leqslant \liminf_{n\to\infty} \{\lambda_{m(n),n}^{(k(n))}\}^{1/n} \leqslant \limsup_{n\to\infty} \{\lambda_{m(n),n}^{(k(n))}\}^{1/n} \leqslant \frac{1}{2^{1-\delta}}. \tag{3.7}$$

*Proof.* First, set $\nu(n) = \max\{0; k(n) - (n+1)\}$, so that $0 \leqslant \nu(n) \leqslant n$. From the integral representation in (2.4), the Padé approximant $R_{\nu(n),n}(x)$ to $e^{-x}$ evidently satisfies

$$| R_{\nu(n),n}(x) - e^{-x} | = \mathcal{O}(| x |^{n+\nu(n)+1}), \qquad | x | \to 0,$$

for each $n \geqslant 0$, and hence, from the definition of $\nu(n)$,

$$| R_{\nu(n),n}(x) - e^{-x} | = \mathcal{O}(| x |^{k(n)}), \qquad | x | \to 0,$$

for each $n \geqslant 0$. Thus, $R_{\nu(n),n} \in \hat{\pi}_{\nu(n),n}^{(k(n))} \subset \hat{\pi}_{m(n),n}^{(k(n))}$ for any integer $m(n)$ with $\nu(n) \leqslant m(n) \leqslant n$. Hence, by definition,

$$\lambda_{m(n),n}^{(k(n))} \leqslant \eta_{\nu(n),n}, \qquad \text{for each} \quad n \geqslant 0.$$

But using (3.4), we have that

$$\{\lambda_{m(n),n}^{(k(n))}\}^{1/n} \leqslant \frac{1}{2^{1-(\nu(n)/n)}},$$

so that applying the hypothesis of (3.6) establishes the last inequality of (3.7). On the other hand, Newman [8] has shown that for *any* polynomials $p$, $q \in \hat{\pi}_n$,

$$\sup_{0 \leqslant x < \infty} \left| e^{-x} - \frac{p(x)}{q(x)} \right| > \frac{1}{(1280)^{n+1}},$$

which establishes the first inequality of (3.7). Q.E.D.

We remark that a stronger result, analogous to (3.7), can be similarly established from the inequality (2.10).

## 4. The Convergence of Particular Padé Approximants to $e^{-z}$ on Unbounded Regions

In this section, we shall be concerned with the convergence, in the uniform norm, of particular Padé approximants to $e^{-z}$ on unbounded sets in the complex plane which are symmetric with respect to the positive ray $0 \leqslant x < \infty$. To begin, let $s_n(z) := \sum_{k=0}^{n} z^k/k!$ denote the familiar $n$-th partial sum of $e^z$. Then, it is clear from (2.2)–(2.3) that the $(0, n)$-th Padé approximant $R_{0,n}(z)$ of $e^{-z}$ is given by

$$R_{0,n}(z) = \frac{1}{s_n(z)}. \tag{4.1}$$

Thus, the poles of the Padé approximant $R_{0,n}$ are the zeros of $s_n$. It is further known that the parabolic region $T$ in the complex plane, defined by

$$T \equiv \{z = x + iy : x \geqslant 0 \quad \text{and} \quad |y| \leqslant dx^{1/2}\}, \tag{4.2}$$

where

$$d < 0.863\,369\,712, \tag{4.3}$$

contains no zeros of any $s_n$, i.e., $1/s_n$ is analytic in $T$ for all $n$ sufficiently large. That such a parabolic region with this property could exist was first indicated by the numerical results of Iverson [4], and the existence of this region was later established[1] by Newman and Rivlin [9].

The special case $\nu = 0$ of (2.8) of Theorem 2.3, coupled with (4.1), implies that

$$\left\| e^{-x} - \frac{1}{s_n(x)} \right\|_{L_\infty[0,\infty]} \leqslant \frac{1}{2^n} \tag{4.4}$$

for all $n \geqslant 0$, and moreover, from Theorem 1 of Meinardus and Varga [6], we have that

$$\lim_{n\to\infty} \left\{ \left\| e^{-x} - \frac{1}{s_n(x)} \right\|_{L_\infty[0,\infty]} \right\}^{1/n} = \frac{1}{2}. \tag{4.5}$$

It is natural to ask if the sequence $\{1/s_n\}_{n=1}^{\infty}$ converges geometrically to $e^{-x}$ on some larger set in the complex plane, especially when we know that $1/s_n$ is analytic in the parabolic region $T$ of (4.2), for all $n$. That this is so is

[1] Strictly speaking, the above-mentioned property of $T$, as stated in [9], does not follow completely from results of [9], but depends additionally on a subsequent note by Newman and Rivlin [10].

established in the following result. For added notation, if $S$ is any set in the complex plane and $f$ is defined on $S$, we write

$$\|f\|_{L_\infty(n)} = \sup\{|f(z)| : z \in S\}.$$

THEOREM 4.1. *Let $g$ be a positive continuous function on* $[0, +\infty)$ *which satisfies*

$$\lim_{x\to+\infty} \frac{g(x)}{(x)^{1/2}} = d^*, \qquad d^* \geqslant 0, \tag{4.6}$$

*and let* $G = \{z = x + iy : x \geqslant 0$ *and* $|y| \leqslant g(x)\}$. *If* (*cf.* (4.2))

$$d^* < d\left(\frac{(2)^{1/2} - 1}{(2)^{1/2} + 1}\right), \qquad \text{e.g.,} \qquad d^* < 0.184\ 130\ 824, \tag{4.7}$$

*then the sequence* $\{1/s_n\}_{n=1}^{\infty}$ *converges geometrically to* $e^{-z}$ *on* $G$. *In particular, if* $d^*$ *of* (4.6) *is positive, then*

$$\limsup_{n\to\infty} \left\{\left\| e^{-z} - \frac{1}{s_n}\right\|_{L_\infty(G)}\right\}^{1/n} \leqslant \frac{1}{2}\left(\frac{d + d^*}{d - d^*}\right)^2 < 1, \tag{4.8}$$

*while if* $d^* = 0$, *then*

$$\lim_{n\to\infty} \left\{\left\| e^{-z} - \frac{1}{s_n}\right\|_{L_\infty(G)}\right\}^{1/n} = \frac{1}{2}. \tag{4.9}$$

*Proof.* By way of construction, it is possible from (4.7) to choose positive numbers $d_0$ and $d_1$ such that $d^* < d_0 < d_1 < d$ and such that $\frac{1}{2}\{(d_1 + d_0)/(d_1 - d_0)\}^2 < 1$. With these positive numbers, the sets $T_i$ are defined:

$$T_i \equiv \{z = x + iy : x \geqslant 0 \quad \text{and} \quad |y| \leqslant d_i x^{1/2}\}, \qquad i = 0, 1,$$

and hence $T_0 \subset T_1$. Next, it is clear from (4.6) that there is a finite $\sigma \geqslant 0$ such that the subset $G_\sigma \equiv \{z = x + iy\colon z \in G$ and $x \geqslant \sigma\}$ of $G$ satisfies

$$G_\sigma \subset T_0.$$

Next, since the zeros of the $s_n$'s have no finite limit point, i.e., if $\{z_j^{(n)}\}_{j=1}^n$ denote the zeros of $s_n$, then $\lim_{n\to\infty} \{\min_{1\leqslant j\leqslant n} |z_j^{(n)}|\} = +\infty$, then for all $n$ sufficiently large, say $n \geqslant n_0$, each $s_n$ is free of zeros in the sets $T_0$, $T_1$, and $G$.

Continuing our construction, for each $t \geqslant 0$ and each $\beta > 0$, let $m(t, \beta)$ be the interval $[t - \beta t^{1/2}, t + \beta t^{1/2}]$ of the real axis. For $t \geqslant \beta^2$, $m(t, \beta)$ lies entirely on the nonnegative axis. Next, for each $\mu > 1$, let $m_\mu(t, \beta)$ denote the level curve of $m(t, \beta)$ in the complex plane, i.e., $m_\mu(t, \beta)$ is an ellipse given by

$$m_\mu(t, \beta) \equiv \left\{z = x + iy : \frac{(x - t)^2}{a^2} + \frac{y^2}{b^2} = 1\right\}, \tag{4.10}$$

where

$$a = a(t, \beta, \mu) = \frac{\beta t^{1/2}}{2}(\mu + \mu^{-1}), \qquad \text{and} \qquad b = b(t, \beta, \mu) = \frac{\beta t^{1/2}}{2}(\mu - \mu^{-1}). \tag{4.11}$$

For each $t \geq \beta^2$, we seek the largest value of $\mu \geq 1$ such that $m_\mu(t, \beta) \subseteq T_1$. This value of $\mu$, which we call $A_1 = A_1(t, \beta, d_1)$ is obtained when $m_\mu(t, \beta)$ is tangent to the parabola $y^2 = d_1^2 x$ which defines $T_1$. In particular, as is readily shown, for $\beta^2 \leq t \leq \beta^2 M$ where $M = 1 + d_1^2/2\beta^2$, $A_1$ is obtained by making $m_\mu(t, \beta)$ tangent to $T_1$ at the origin, and $A_1$ is given in this case by

$$A_1 = \frac{t^{1/2}}{\beta} + \left(\frac{t}{\beta^2} - 1\right)^{1/2}, \qquad \beta^2 \leq t \leq \beta^2 M. \tag{4.12}$$

For $t > \beta^2 M$, $m_\mu(t, \beta)$ will have exactly two points of intersection with $T_1$, i.e.,

$$(x - t)^2/a^2 + d_1^2 x/b^2 = 1$$

will have exactly one (nonnegative) root for $x$, precisely when the discriminant of the above quadratic in $x$, equals zero:

$$\{2t - a^2 d_1^2/b^2\}^2 - 4(t^2 - a^2) = 0,$$

or equivalently, solving for $b^2$,

$$b^2 = \frac{d_1^2}{2}\{t + (t^2 - a^2)^{1/2}\}.$$

Thus, with (4.11), the largest value of $\mu \geq 1$, i.e., $A_1$, for which $m_\mu(t, \beta) \subseteq T_1$ satisfies

$$\left(A_1 - \frac{1}{A_1}\right)^2 = 2\left(\frac{d_1}{\beta}\right)^2 \left\{1 + \left(1 - \frac{1}{4u^2}\left(A_1 + \frac{1}{A_1}\right)^2\right)^{1/2}\right\}, \qquad u^2 = \frac{t}{\beta^2} > M, \tag{4.12'}$$

which gives rise to a polynomial equation in $A_1$ of degree 6. It is apparent from (4.12) and (4.12′) that $A_1 = A_1(t, \beta, d_1)$ is in reality a function of $u = t^{1/2}/\beta$ and $d_1/\beta$, and we also write $A_1 = A_1(u, d_1/\beta)$. It is also clear from (4.12′) that $A_1$ is a continuous strictly increasing function of $u$. Next, to obtain an upper bound for $A_1$, one sees geometrically that forcing the ellipse $m_\mu(t, \beta)$ to intersect the curve $y = d_1 t^{1/2}$ in the particular point $(t, b)$ must give an upper bound for $A_1$. Thus, $b = d_1 t^{1/2} = \beta t^{1/2}(\hat{u} - 1/\hat{u})/2$ implies $A_1 < \hat{u}$, or equivalently

$$A_1\left(u, \frac{d_1}{\beta}\right) < \left(\frac{d_1}{\beta}\right) + \left(1 + \left(\frac{d_1}{\beta}\right)^2\right)^{1/2} \text{ for all } u \geq 1.$$

Hence, $A_1(u, d_1/\beta)$ is bounded, for fixed $d_1/\beta$, as $u \to +\infty$. Using this fact, it follows from (4.12′) that the above upper bound is asymptotically sharp:

$$\lim_{u\to+\infty} A_1\left(u, \frac{d_1}{\beta}\right) = \alpha_1\left(\frac{d_1}{\beta}\right) = \left(\frac{d_1}{\beta}\right) + \left(1 + \left(\frac{d_1}{\beta}\right)^2\right)^{1/2}. \quad (4.13)$$

Similarly, if $A_0(t, \beta, d_0) = A_0(u, d_0/\beta)$ denotes the largest value of $\mu \geqslant 1$ such that $m_\mu(t, \beta)$ is contained in $T_0$ for all $t \geqslant \beta^2$, the argument above directly gives

$$\lim_{u\to+\infty} A_0\left(u, \frac{d_0}{\beta}\right) = \alpha_0\left(\frac{d_1}{\beta}\right) = \left(\frac{d_0}{\beta}\right) + \left(1 + \left(\frac{d_0}{\beta}\right)^2\right)^{1/2}. \quad (4.13')$$

Next, it is straightforward to deduce from (4.13) and (4.13′) that

$$\lim_{\beta\to+\infty} \left\{\frac{\alpha_1(d_1/\beta)\,\alpha_0(d_0/\beta) - 1}{\alpha_1(d_1/\beta) - \alpha_0(d_0/\beta)}\right\} = \frac{d_1 + d_0}{d_1 - d_0} < (2)^{1/2}, \quad (4.14)$$

the last inequality following from our choice of $d_0$ and $d_1$.

Now, with the inequality of (4.4), we have

$$\left|\frac{1}{s_{n+1}(x)} - \frac{1}{s_n(x)}\right| \leqslant \left|\frac{1}{s_{n+1}(x)} - e^{-x}\right| + \left|e^{-x} - \frac{1}{s_n(x)}\right|$$
$$\leqslant \frac{1}{2^{n+1}} + \frac{1}{2^n} = \frac{3}{2^{n+1}},$$

for any $x \geqslant 0$ and any $n \geqslant 0$. In particular, for any $t \geqslant \beta^2$ (so that $m(t, \beta)$ lies entirely on the nonnegative axis),

$$\left|\frac{1}{s_{n+1}(x)} - \frac{1}{s_n(x)}\right| \leqslant \frac{3}{2^{n+1}}, \qquad x \in m(t, \beta),\ t \geqslant \beta^2, \qquad n \geqslant 0.$$

In addition, we know that the rational function $(1/s_{n+1} - 1/s_n) \in \pi_{n+1,2n+1}$ has, for any $n \geqslant n_0$ all its poles outside of $T_1$. Then, applying *Walsh's Lemma* (cf. [16; Eq. (41), p. 250]) to this rational function on the set $m(t, \beta)$ yields

$$\left|\frac{1}{s_{n+1}(z)} - \frac{1}{s_n(z)}\right| \leqslant \frac{3}{2^{n+1}} \left\{\frac{A_1(t, \beta, d_1)\, A_0(t, \beta, d_0) - 1}{A_1(t, \beta, d_1) - A_0(t, \beta, d_0)}\right\}^{2n+1},$$

for all $z \in \bar{m}_{A_0}(t, \beta)$, $t \geqslant \beta^2$, $n \geqslant n_0$, where $\bar{m}_\mu(t, \beta)$ denotes all points $z$ on or inside $m_\mu(t, \beta)$ i.e.,

$$\bar{m}_\mu(t, \beta) = \left\{z = x + iy : \frac{(x-t)^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\right\}.$$

Hence, given any $\epsilon > 0$ sufficiently small, so that

$$((d_1 + d_0)/(d_1 - d_0) + \epsilon) < (2)^{1/2},$$

it follows from (4.13)–(4.14) that there is a $\tilde{\beta}$ and a $\tilde{u}$ sufficiently large so that

$$\left| \frac{1}{s_{n+1}(z)} - \frac{1}{s_n(z)} \right| \leqslant \frac{3}{2^{n+1}} \left\{ \frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right\}^{2n+1}, \tag{4.15}$$

for all $n \geqslant n_0 - 1$, for all $z \in \bar{m}_{A_0}(t, \tilde{\beta})$, and for all $t \geqslant \tilde{\beta}^2 \tilde{u}^2$. Thus, since

$$\left| \frac{1}{s_{n+r}(z)} - \frac{1}{s_n(z)} \right| \leqslant \sum_{j=0}^{r-1} \left| \frac{1}{s_{n+j+1}(z)} - \frac{1}{s_{n+j}(z)} \right|$$

for any $r \geqslant 1$, then applying the inequality of (4.15) in the above sum and summing the resultant geometric series gives

$$\left| \frac{1}{s_{n+r}(z)} - \frac{1}{s_n(z)} \right| \leqslant \frac{3\gamma^{2n+1}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \qquad \gamma = \left[ \frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right].$$

Consequently, letting $r \to \infty$,

$$\left| e^{-z} - \frac{1}{s_n(z)} \right| \leqslant \frac{3\gamma^{2n+1}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \; z \in \bar{m}_{A_0}(t, \beta),\; t \geqslant \beta^2 u^2,\; n \geqslant n_0. \tag{4.16}$$

Now, by construction, the closed ellipses $\bar{m}_{A_0}(t, \beta)$ trace out the set $T_0$, i.e., for every $\beta > 0$,

$$\bigcup_{t \geqslant \beta^2} \{\bar{m}_{A_0}(t, \beta)\} = T_0 .$$

Hence, the set $\bigcup_{t \geqslant \tilde{\beta}^2 \tilde{u}^2} \{\bar{m}_{A_0}(t, \tilde{\beta})\}$ can be expressed as $T_0 - C$, where $C = C(\epsilon)$ is some compact set in the complex plane. Thus, (4.16) can be equivalently expressed as

$$\left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(T_0 - C)} \leqslant \frac{3\gamma^{2n+1}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \qquad n \geqslant n_0 .$$

Recalling that the set $G$ of Theorem 4.1 is a subset of $T_0 - C$ with the exception of some compact set $C'$, this implies that

$$\limsup_{n \to \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G - C')} \right\}^{1/n} \leqslant \frac{1}{2} \left[ \frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right]^2. \tag{4.17}$$

On the other hand, for any compact set $C$,

$$\lim_{n \to \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(C)} \right\}^{1/n} = 0. \tag{4.18}$$

To see this, define $0 < \delta = \inf\{|e^{-z}| : z \in C\}$, and $\rho = \sup\{|z| : z \in C\}$. Because of the uniform convergence of $s_n$ to $e^z$ on $C$, then $\delta/2 \leqslant |s_n(z)|$ for all $z \in C$, all $n \geqslant n_1$. Thus, for $n \geqslant \max\{\rho - 2, n_1\}$,

$$\left| e^{-z} - \frac{1}{s_n(z)} \right| = \frac{|s_n(z) - e^z|}{|e^z \cdot s_n(z)|} \leqslant \frac{2}{\delta^2} |s_n(z) - e^z| = \frac{2}{\delta^2} \left| \sum_{k=n+1}^{\infty} z^k/k! \right|$$
$$\leqslant \frac{2}{\delta^2} \sum_{k=n+1}^{\infty} \rho^k/k! \leqslant \frac{2(n+2)\,\rho^{n+1}}{\delta^2(n+1)!\,(n+2-\rho)},$$

for all $z \in C$. Thus, using Stirling's formula, (4.18) follows. Hence, combining (4.17) and (4.18), we deduce that

$$\limsup_{n\to\infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leqslant \frac{1}{2} \left[ \frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right]^2. \tag{4.19}$$

Thus, letting both $\epsilon \to 0$ in (4.19) yields

$$\limsup_{n\to\infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leqslant \frac{1}{2} \left( \frac{d_1 + d_0}{d_1 - d_0} \right)^2.$$

Finally letting $d_1 \to d$ and $d_0 \to d^*$ in the above expression then establishes

$$\limsup_{n\to\infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leqslant \frac{1}{2} \left( \frac{d + d^*}{d - d^*} \right)^2 < 1,$$

the desired result of (4.8). Of course, if $d^* = 0$, then

$$\limsup_{n\to\infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leqslant \frac{1}{2}.$$

But as $[0, +\infty)$ is a subset of $G$, it follows from (4.5) and the above inequality, that

$$\frac{1}{2} \leqslant \limsup_{n\to\infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leqslant \limsup_{n\to\infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\}^{1/n} \leqslant \frac{1}{2},$$

whence $\liminf_{n\to\infty} \{\| e^{-z} - 1/s_n \|_{L_\infty(G)}\}^{1/n} = \frac{1}{2}$, the desired result of (4.9). Q.E.D.

As a special case of Theorem 4.1, we have

COROLLARY 4.2. *For any semi-infinite strip*

$$I_\tau = \{z = x + iy : x \geqslant 0, |y| \leqslant \tau\},$$

*where* $0 \leqslant \tau < \infty$,

$$\lim_{n\to\infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(I_\tau)} \right\}^{1/n} = \frac{1}{2}.$$

It is again natural to ask if the geometric convergence of (4.8)–(4.9) of Theorem 4.1 holds for similar unbounded domains in the complex plane, for other Padé approximations of $e^{-z}$. Such a result, which would extend Theorem 3.2 to larger sets in the complex plane, of course depends on a precise knowledge of the location of the poles of other Padé approximations of $e^{-z}$, which seems not to be known in the general case. On the other hand, the uniform convergence of Padé approximants to $e^{-z}$ on $[0, +\infty)$ of Theorem 3.1 can be similarly extended to larger sets in the complex plane for particular Padé approximations, as we now show.

THEOREM 4.3. *Given any* $\delta$ *with* $0 < \delta \leqslant \pi/2$, *the sequences*

$$\{R_{n-1,n}(z)\}_{n=1}^{\infty} \qquad \text{and} \qquad \{R_{n-2,n}(z)\}_{n=2}^{\infty}$$

*converge uniformly to* $e^{-z}$ *on the sector* $S_\delta = \{z = re^{i\theta} : |\theta| \leqslant \pi/2 - \delta\}$.

*Proof.* It was originally shown by Birkhoff and Varga [1] that all the Padé approximants $R_{n,n}(z)$ of $e^{-z}$ are analytic in the right-half plane Re $z \geqslant 0$, and are bounded in modulus there by unity. More recently, Ehle [3] has extended both of these results to $\{R_{n-1,n}(z)\}_{n=1}^{\infty}$ and $\{R_{n-2,n}(z)\}_{n=2}^{\infty}$. Dealing for definiteness with $\{R_{n-1,n}(z)\}_{n=1}^{\infty}$, we thus have that each

$$f_n(z) = e^{-z} - R_{n-1,n}(z)$$

is analytic in the open first quadrant $S = \{z = x + iy : x > 0 \text{ and } y > 0\}$, and that $\sup\{|f_n(z)| : z \in S\} \leqslant 2$, for all $n \geqslant 1$. Since the boundary of $S$ consists of the rays $\gamma_1 = \{z = x + iy : x \geqslant 0, y = 0\}$ and

$$\gamma_2 = \{z = x + iy : x = 0, y \geqslant 0\},$$

the harmonic measure $w(z)$ of $\gamma_1$ with respect to $S$, defined as a function which is harmonic and bounded in $S$ and for which $w(z) = 1$ for all $z \in \text{int } \gamma_1$ and $w(z) = 0$ for all $z \in \text{int } \gamma_2$, is obviously given by

$$w(z) = 1 - \frac{2}{\pi} \arg z. \tag{4.20}$$

Then, by the *Nevanlinna Two-Constants Theorem* (cf. [7, p. 41]), if

$$M_i = \sup\{|f_n(z)| : z \in \text{int } \gamma_i\}, \qquad i = 1, 2,$$

then

$$|f_n(z)| \leqslant M_1^{w(z)} \cdot M_2^{1-w(z)}, \qquad \text{for all} \quad z \in S. \tag{4.21}$$

Strictly speaking the Two-Constants Theorem is stated for *bounded* domains. Therefore, the validity of (4.21) follows by considering an appropriate conformal mapping of $S$.

Now since $M_1 = \eta_{n-1,n}$ (cf. (2.5)), and $M_2 \leqslant 2$, it follows from (4.20) and (4.21) that

$$|f_n(z)| \leqslant \eta_{n-1,n}^{1-(2/\pi)\arg z} \cdot 2^{(2/\pi)\arg z}, \qquad \text{for all} \quad z \in S.$$

Thus, as $\arg(z) < \pi/2$ in $S$,

$$|f_n(z)| \leqslant 2\eta_{n-1,n}^{1-(2/\pi)\arg(z)}, \qquad \text{all} \quad z \in S.$$

Now, from Proposition 2.4, there exists an $n_0 > 0$ such that $\eta_{n-1,n} < 1$ for all $n \geqslant n_0$. Thus, restricting $z$ to be in the sector $S_\delta^+ := \{z = re^{i\theta} : 0 \leqslant \theta \leqslant \pi/2 - \delta\}$ where $0 < \delta \leqslant \pi/2$, then

$$|f_n(z)| \leqslant 2 \cdot \eta_{n-1,n}^{(2/\pi)\delta}, \qquad \text{all} \quad z \in S_\delta^+,$$

and, as the same result evidently holds for the reflected sector $S_\delta^- = \{z = re^{i\theta} : -(\pi/2 - \delta) \leqslant \theta \leqslant 0\}$, we have

$$\| e^{-z} - R_{n-1,n} \|_{L_\infty(S_\delta)} \leqslant 2\eta_{n-1,n}^{(2/\pi)\delta}.$$

Thus, since $\eta_{n-1,n} \to 0$ as $n \to \infty$ from Proposition 2.4, then $\{R_{n-1,n}(z)\}_{n=1}^{\infty}$ converges uniformly to $e^{-z}$ on $S_\delta$, the same conclusion being true also for $\{R_{n-2,n}(z)\}_{n=2}^{\infty}$. Q.E.D.

## Acknowledgment

The authors sincerely thank Mrs. Grace Bush of Kent State University and Professor Arthur Price of the University of South Florida for their generous assistance in carrying out various computer calculations.

## References

1. G. Birkhoff and R. S. Varga, Discretization errors for well-set Cauchy problems. I, *J. Math. and Phys.* **44** (1965), 1–23.
2. W. J. Cody, G. Meinardus, and R. S. Varga, Chebyshev rational approximations to $e^{-x}$ in $[0, +\infty)$ and applications to heat-conduction problems, *J. Approximation Theory*, **2** (1969), 50–65.
3. B. L. Ehle, A stable and Padé approximations to the exponential, *SIAM J. Math. Anal.* **4** (1973), 671–680.
4. K. E. Iverson, The zeros of the partial sums of $e^z$, *Math. Tables Aids Comp.* **7** (1953), 165–168.

5. J. D. LAWSON, Order constrained Chebyshev rational approximation, *Math. Comp.*, to appear.
6. G. MEINARDUS AND S. VARGA, Chebyshev rational approximations to certain entire functions in $[0, +\infty)$, *J. Approximation Theory*, **3** (1970), 300–309.
7. R. NEVANLINNA, "Analytic Functions," Springer Verlag, New York, 1970.
8. D. J. NEWMAN, Rational approximation to $e^{-x}$, *J. Approximation Theory* **10** (1974), 301–303.
9. D. J. NEWMAN AND T. J. RIVLIN, "The zeros of the partial sums of the exponential function," *J. Approximation Theory*, **5** (1972), 405–412.
10. D. J. NEWMAN AND T. J. RIVLIN, The zeros of the partial sums of the exponential function, *J. Approximation Theory*, to appear.
11. H. PADÉ, "Sur la représentation approchée d'une fonction par des fractions rationnelles," *Thesis*, *Ann. de l'Éc. Nor.* **9**, 1892.
12. O. PERRON, "Die Lehre von den Kettenbrücken. II," Teubner, Leipzig; 1929; reprint, Chelsea, New York.
13. A. SCHÖNHAGE, Zur rationalen Approximierbarkeit von $e^{-x}$ über $[0, +\infty)$, *J. Approximation Theory* **7** (1973), 395–398.
14. R. S. VARGA, On higher order stable implicit methods for solving parabolic partial differential equations, *J. Math. and Phys.* **40** (1961), 220–231.
15. R. S. VARGA, "Matrix Iterative Analysis," Prentice-Hall, Englewood Cliffs, NJ, 1962.
16. J. L. WALSH, Interpolation and approximation by rational functions in the complex domain, *in* "Colloquium Publication," Vol. 20, Amer. Math. Soc., Providence, RI, Fifth ed., 1969.